FINANCIAL MANAGEMENT USING INPUT-OUTPUT SWITCHING OPTIONS 10 January 2020

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Abstract

There are at least four types of basic output, input, or output-input switching options that are useful in capital budgeting for choosing outputs or inputs, and for start-up and shut-down decisions. These basic two factor real option models now have analytical solutions, with easy to use Excel formulas. Extensions enable the financial manager to evaluate the effect of changing input or output volatilities, and correlations, on both the input or output level that justifies immediate action, and also the real option value before that action. Empirical results show that the sensitivity to changes in expected volatilities and correlations are not always intuitive, and depend on the particular type of switching option. Case studies on practical application to drilling for either natural gas or natural gas liquids in Appalachia are available from the authors.

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1 Introduction

When is the right time for an operator or investor to switch between two possible outputs or inputs in order to maximise value when switching costs are taken into account? Which factors should be monitored in making these decisions? How much should an investor pay for such a flexible opportunity or for such an alert manager? What are the strategy implications for the manager, investor and possibly for policy makers?

The traditional approach to determine switching boundaries between two operating modes is to discount future cash flows and use Jevons-Marshallian present value of the output-input plus switching cost as the threshold justifying immediate action. This methodology does not fully capture the option value which may arise due to the uncertainty in future input or output prices. The value of waiting to gain more information on future price or cost developments, and consequently on the optimal switching thresholds, can be viewed in a real options framework.

Single switching models are appropriate for instances where the switch is irreversible, no exit is feasible or likely over a long period, which could cover many types of infrastructure, bridges, rail, road, drilling expenditures, and for the shut-down option, where closed down facilities cannot be reopened.

This study presents four basic two factor switching option models: (i) switching to the highest price output; (ii) switching from an operating state to an abandoned state, when both output price and input cost are stochastic; (iii) switching from an idle state to a permanent operating state, when both output price and input cost are stochastic; and (iv) switching to the lowest cost input. Adapting the Heydari et al. (2012), and Støre et al. (2018) analytical solutions, the model for the best of two outputs is originally from Dockendorf and Paxson (2013), for input-output startup investments (similar to renewals) from Adkins and Paxson (2006, 2011b), input-output shut down from Adkins and Paxson (2012), and for two inputs from Heydari et al. (2012) and Adkins and

Paxson (2011a). Note these stochastic price variables could be viewed in terms of revenues and costs (which include the quantity of production), or similar suitable factors such as votes or participants, where economic value is not considered.

Conceptually, the switch between two volatile assets or commodities can be modelled as an exchange option. Margrabe (1978) and McDonald and Siegel (1986) model European and American perpetual exchange options, respectively, which are linear homogeneous in the underlying stochastic variables. Adkins and Paxson (2006, 2011b), Gahungu and Smeers (2011), and Rohlfs and Madlener (2011) present quasi-analytical or analytical solutions to switching options, where two-factor functions are not homogeneous of degree one, and thus dimension reducing techniques are not available.

With some simplifying assumptions, such as single irreversible choices and constant correlation of stochastic factors, the objective is to find the prospective output or input that justifies immediate action, and the real option value (ROV) of the switching opportunity. We provide unique analytical formations for input-output switching, and new simplified formations for real option values for the other switching options, plus numerical illustrations of solving the partial differential equations in all cases. The results show that vegas (changes in the threshold and/or real option value as volatility increases) are often negative then positive (or vice versa) with positive correlations, consistent with quasi-analytical solutions shown by various authors. Also, the chi's (changes in the threshold and/or real option value as correlation increases) are usually negative, but sometimes positive, depending on the particular type of switching option.

The next section presents the basic value matching conditions, and basic analytical solutions, for four single switching opportunities between two outputs with uncertain prices, taking into account switching costs and operating costs, or two inputs, or inputs-outputs. All of these models assume that the two factors follow geometric Brownian motions, possibly correlated, as in the quasi-analytical and analytical solutions, with most derivations relegated to the Appendix.

2 Value Matching Conditions and Analytical Solutions

2.1 Value Matching Conditions

The Output-Output (OO), Input-Output ShutDown (IO-SD), Input-Output StartUp (IO-SU), and Input-Input (II) single switch options have value matching conditions, and analytical solutions, which may appear similar, except for signs and power parameter subscripts, where x is the initial output in OO, the output in IO-SU and IO-SD, and the initial input in II. The power parameters β_1 and β_2 are the positive and negative roots of the characteristic root equations shown below.

OO x=initial output, y=alternative output

$$A\hat{x}^{\beta_2}\,\hat{y}^{\beta_1}+\frac{\hat{x}}{\delta_x}-\frac{c_x}{r}-\{\frac{\hat{y}}{\delta_y}-\frac{c_y}{r}-SO\}=0$$

Negative β_2 is the power parameter for x since a decrease in the initial output price x favours a switch to a higher output y. The first three terms are the ROV and operating value of x, and the last three terms constitute the NPV of the switch to output y at the threshold.

IO-SD x=output, y=input

$$A\hat{x}^{\beta_2}\,\hat{y}^{\beta_1}+\frac{\hat{x}}{\delta_x}-\frac{\hat{y}}{\delta_y}+\{D\}=0$$

Negative β_2 is the power parameter for x since a decrease in the output price x favours a shutdown with the input y. The last term is the negative NPV (decommissioning cost) of the shutdown at the threshold.

IO-SU x=output, y=input

$$A\hat{x}^{\beta_1}\,\hat{y}^{\beta_2}-\{\frac{\hat{x}}{\delta_x}-\frac{\hat{y}}{\delta_y}-K\}=0$$

Positive β_1 is the power parameter for x since an increase in the output price x favours a startup with the input y. The last three terms constitute the NPV of the startup at the threshold.

II x=initial input, y=alternative input

$$A\hat{x}^{\beta_1}\,\hat{y}^{\beta_2}+\frac{p_x}{r}-\frac{\hat{x}}{\delta_x}-\{\frac{p_y}{r}-\frac{\hat{y}}{\delta_y}-SI\}=0$$

Positive β_1 is the power parameter for x since an increase in the initial input price x favours a switch to a lower input y. The first three terms are the ROV and operating value of x, and the last three terms constitute the NPV of the switch to input y at the threshold.

2.2 Analytical Solutions

In all cases, there are two smooth pasting conditions, each the partial derivative of the value matching condition with respect to $\hat{x}$ and $\hat{y}$, and β_1 and β_2 satisfying the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_2\left(\beta_2-1\right)+\tfrac{1}{2}\sigma_y^2\beta_1\left(\beta_1-1\right)+\rho\sigma_x\sigma_y\beta_2\beta_1+\beta_2\left(r-\delta_x\right)+\beta_1\left(r-\delta_y\right)-r=0 \quad (1)$$

$$\tfrac{1}{2}\sigma_x^2\beta_1\left(\beta_1-1\right)+\tfrac{1}{2}\sigma_y^2\beta_2\left(\beta_2-1\right)+\rho\sigma_x\sigma_y\beta_1\beta_2+\beta_1\left(r-\delta_x\right)+\beta_2\left(r-\delta_y\right)-r=0 \quad (2)$$

The characteristic root equation (1) for OO and IO SD, (2) for IO SU and II together with value matching conditions and two smooth pasting conditions represent a system of 4 equations, while there are 5 unknowns, β_1, β_2, A, $\hat{x}, \hat{y}$. The set of 4 equations can be solved simultaneously deriving $\hat{y}$ assuming $\hat{x}=x$.

OO, IO SD

Following Støre et al. (2018), as simplified in Adkins and Paxson (2018), for OO and IO SD

$$\hat{y}(\hat{x})=\frac{-\beta_1\delta_y\hat{x}}{\beta_2\delta_x} \quad (3)$$

$$A=-\frac{1}{\beta_2\delta_x\hat{x}^{\beta_2-1}\hat{y}^{\beta_1}} \quad (4)$$

$$\frac{\hat{x}}{\delta_x}\frac{\beta_1+\beta_2-1}{\beta_2}+S=0 \quad (5)$$

where S=D for IO SD.

Now there is a system of three equations with four unknowns, $\hat{x}, \hat{y}, \beta_1, \beta_2$.

Assuming that for OO the production costs are the same for x and y, let

$$C(\hat{x})=1+\frac{\delta_x}{\hat{x}}S \quad (6)$$

Note that if S=0, or $\delta_x=0$, this is a linear homogenous problem with a conventional solution. If not, adapting Støre et al. (2018), a solution for $\beta_2(\hat{x})$ is given by:

$$\beta_2(\hat{x}) = \frac{f(\hat{x})}{2g(\hat{x})} - \sqrt{(\frac{f(\hat{x})}{2g(\hat{x})})^2 + \frac{2\delta_y}{g(\hat{x})}} \tag{7}$$

where

$$f(\hat{x}) = \sigma_x^2 - 2(r-\delta_x) - 2\rho\sigma_x\sigma_y + C(\hat{x})(2(r-\delta_y)+\sigma_y^2) \tag{8}$$

and

$$g(\hat{x}) = \sigma_x^2 + \sigma_y^2 C(\hat{x})^2 - 2\rho\sigma_x\sigma_y C(\hat{x}) \tag{9}$$

$$\beta_1 = 1 - \beta_2(\hat{x})C(\hat{x}) . \tag{10}$$

Substituting β_1 and β_2 into (3) and (4) yields the analytical solutions for $\hat{y}(\hat{x})$ and ROV.

The option to switch is:

$$ROV = Ax^{\beta_2} y^{\beta_1} = \frac{-\hat{x}}{\beta_2\delta_x}(\frac{y}{\hat{y}})^{\beta_1} \tag{11}$$

IO SU, II

Adapting Støre et al. (2018), as simplified in Adkins and Paxson (2018), for IO SU and II

$$\hat{y}(\hat{x}) = \frac{-\beta_2\delta_y\hat{x}}{\beta_1\delta_x} \tag{12}$$

$$A = -\frac{1}{\beta_1\delta_x\hat{x}^{\beta_1-1}\hat{y}^{\beta_2}} \tag{13}$$

Assuming that the output for x and y for II is the same, let

$$C(\hat{x}) = 1 - \frac{\delta_x}{\hat{x}}K \tag{14}$$

Note that if K=0, or $\delta_x = 0$, this is a linear homogenous problem with a conventional solution. If not, adapting Støre et al. (2018), and Adkins and Paxson (2012) a solution for $\beta_1(\hat{x})$ is given by:

$$\beta_1(\hat{x}) = \frac{f(\hat{x})}{2g(\hat{x})} + \sqrt{(\frac{f(\hat{x})}{2g(\hat{x})})^2 + \frac{2\delta_y}{g(\hat{x})}} \tag{15}$$

where

$$f(\hat{x}) = \sigma_x^2 - 2(r - \delta_x) - 2\rho\sigma_x\sigma_y + C(\hat{x})(2(r - \delta_y) + \sigma_y^2) \quad (16)$$

and

$$g(\hat{x}) = \sigma_x^2 + \sigma_y^2 C(\hat{x})^2 - 2\rho\sigma_x\sigma_y C(\hat{x}) \quad (17)$$

$$\beta_2 = 1 - \beta_1(\hat{x})C(\hat{x}) . \quad (18)$$

Substituting β_1 and β_2 into (12) and (13) yields the analytical solutions for $\hat{y}(\hat{x})$ and ROV. The option to switch is:

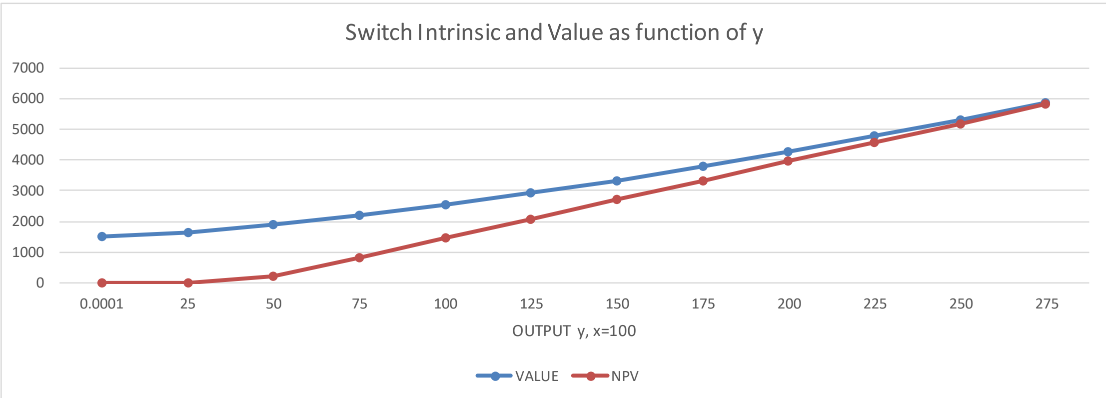
$$ROV = Ax^{\beta_1} y^{\beta_2} = \frac{-\hat{x}}{\beta_1 \delta_x} \left(\frac{y}{\hat{y}}\right)^{\beta_2} \quad (19)$$

3. Sensitivity Illustrations

The common parameter values for all of these figures are shown in the Appendix, for the output switch x=100, y=100, equal convenience yields 4%, r=5%, SO=50, $\sigma_x = .40, \sigma_y = .30, \rho = .5$. For the input-output x=100, y=50, and for the input switching x=50, y=50, for general consistency.

Figure 1

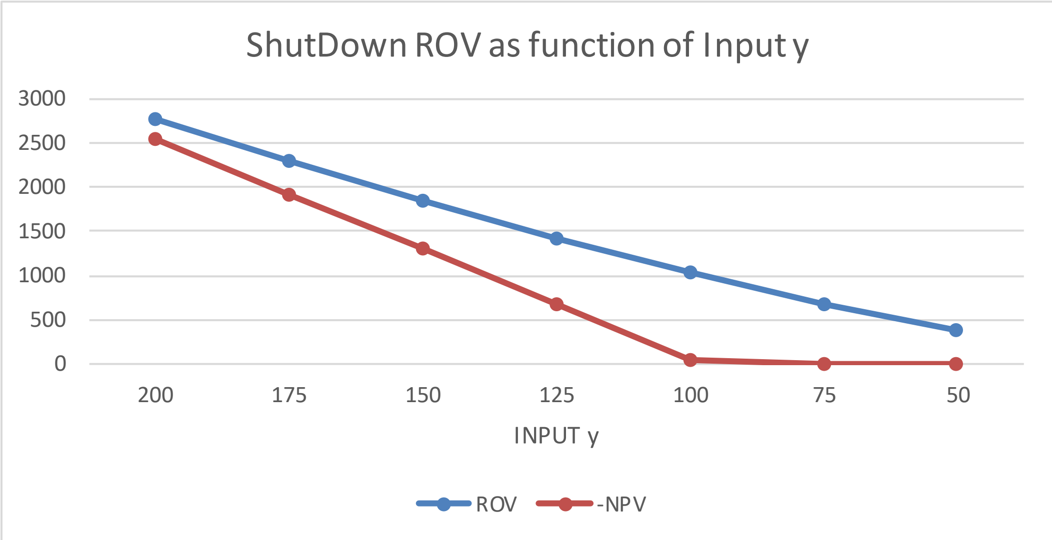
Output Switching ROV and Intrinsic Net Present Value



Operating costs c_y are 50, $NPV = Max[0, \frac{y}{\delta_y} - \frac{c_y}{r} - SO]$.

Figure 2

Output-Input ShutDown ROV and Intrinsic Net Present Value

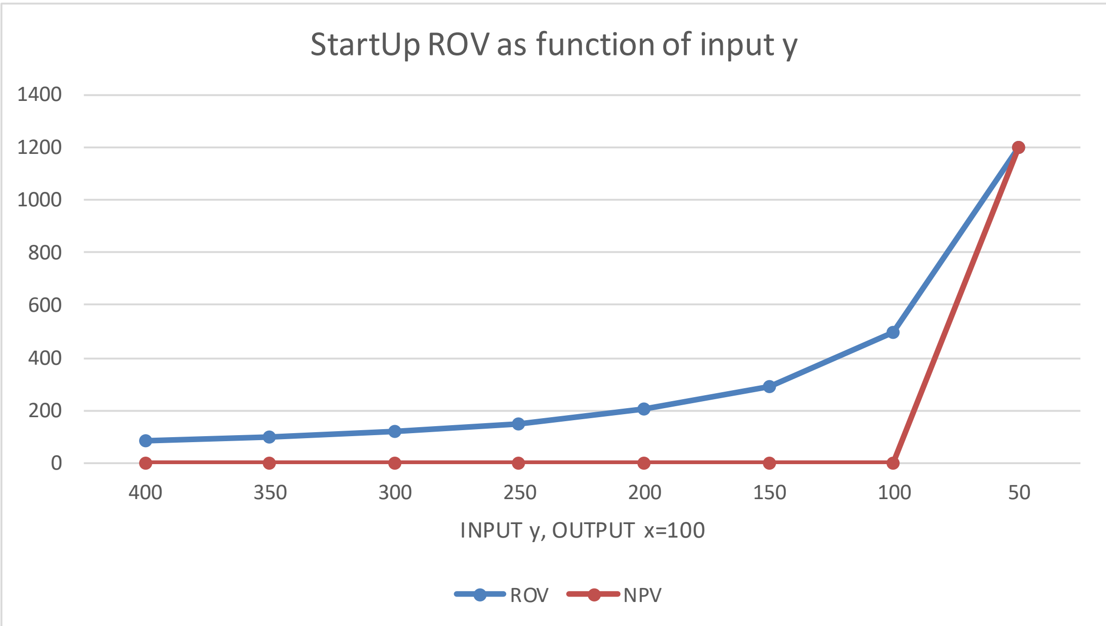


Operating cost =y as input, for comparison -NPV is a construct

$$-NPV = Max[-(\frac{\hat{x}}{\delta_x} - \frac{y}{\delta_y} - D), 0].$$

Figure 3

Output-Input StartUp ROV and Intrinsic Net Present Value

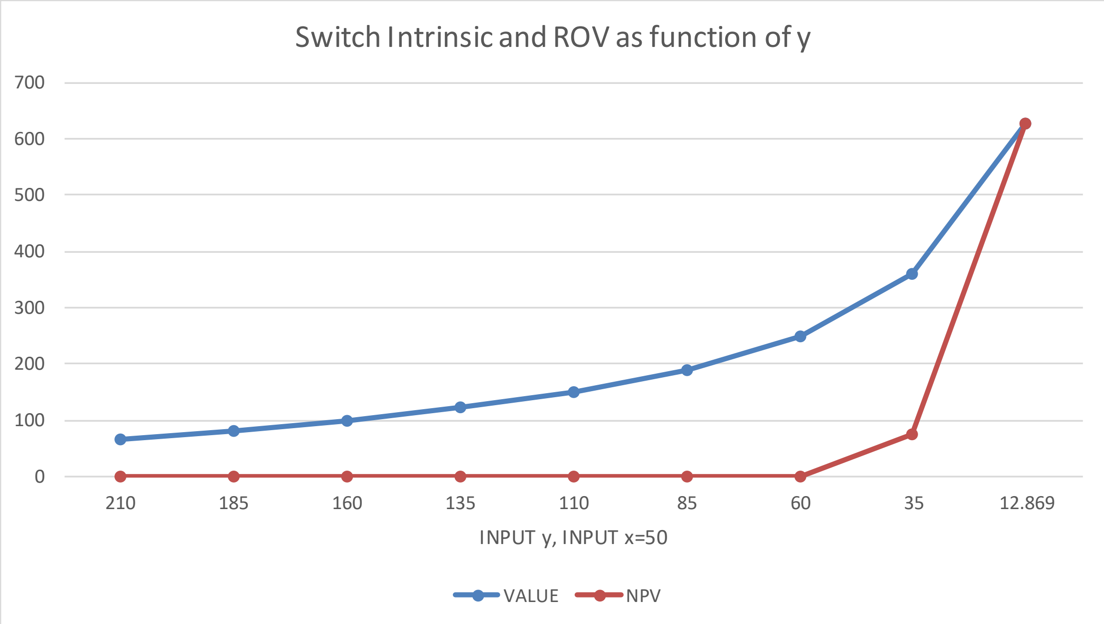


Operating cost =y as input, for comparison NPV is a construct

$$NPV = [Max(\frac{\hat{x}}{\delta_x} - \frac{y}{\delta_y} - K, 0)].$$

Figure 4

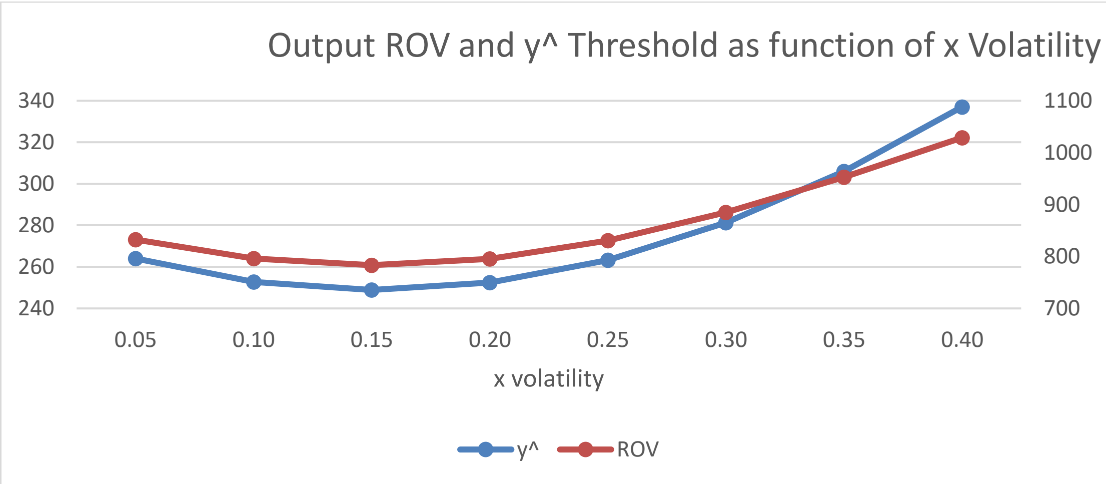
Input-Input ROV and Intrinsic Net Present Value



Output is 100, $NPV = Max[0, \frac{p_y}{r} - \frac{y}{\delta_y} - SI]$.

Figure 5

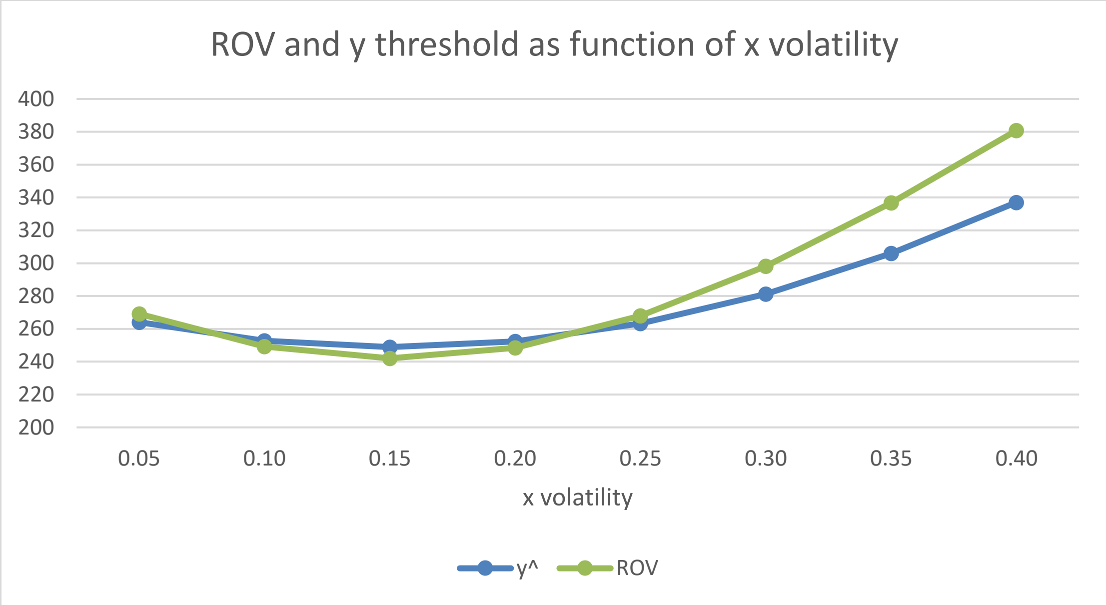
OO Switching ROV & y Threshold Convex Vega, when ρ>0.



If correlation is positive, the threshold $\hat{y}$ and the ROV switching option value (x=100) first decrease as x (or y shown in Appendix A) volatility increases from a low level, and then both increases. Typically, both threshold and real option vegas (sensitivity to increases in volatility) are positive for one factor models, but not in the case of two factor models with these parameter values. However, the base case y threshold is quite high, with 50% correlation and 30% y volatility.

Figure 6

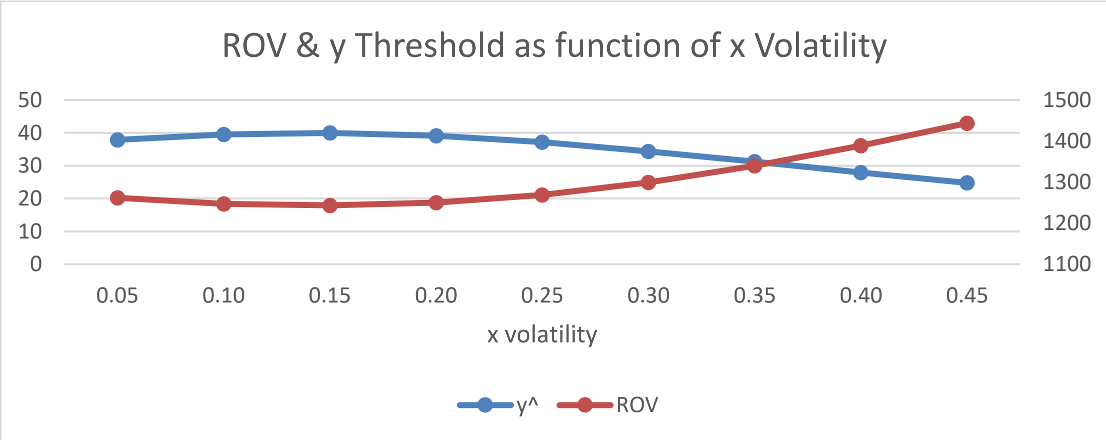
Output-Input ShutDown ROV and y threshold Vegas



IO Shutdown ROV and input y threshold both decline as x or y volatilty increases, then increase, if correlation is positive.

Figure 7

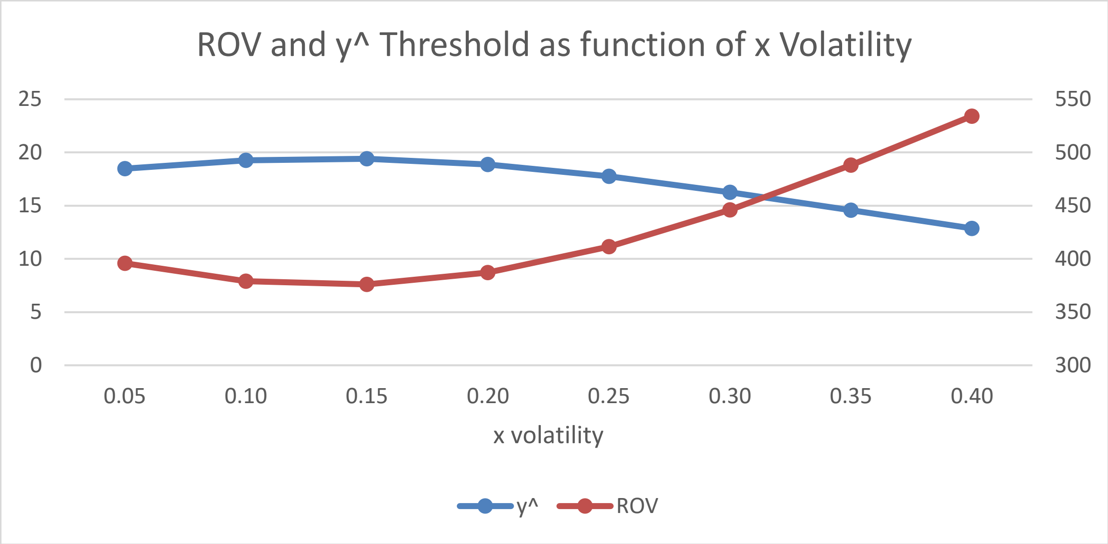
Input-Output StartUp ROV and y Threshold Vegas



IO StartUp ROV decreases and input y threshold increases as x or y volatilty first increases, then increases/decreases, if correlation is positive. If $\rho \leq 0$, ROV vegas are always positive, y threshold vegas always negative.

Figure 8

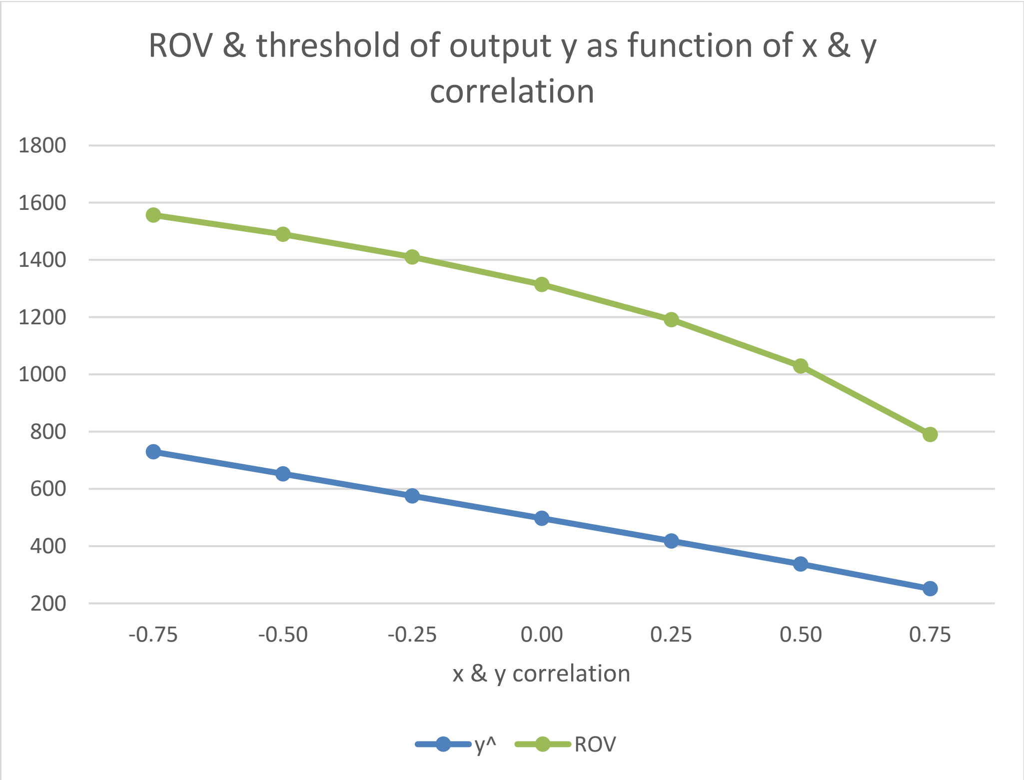
Input-Input ROV and y Threshold Vegas



II switching ROV decreases and y threshold increases as x or y volatilty first increases, then increase/decrease if correlation is positive. If $\rho \leq 0$, ROV vegas are always positive, y threshold vegas always negative.

Figure 9

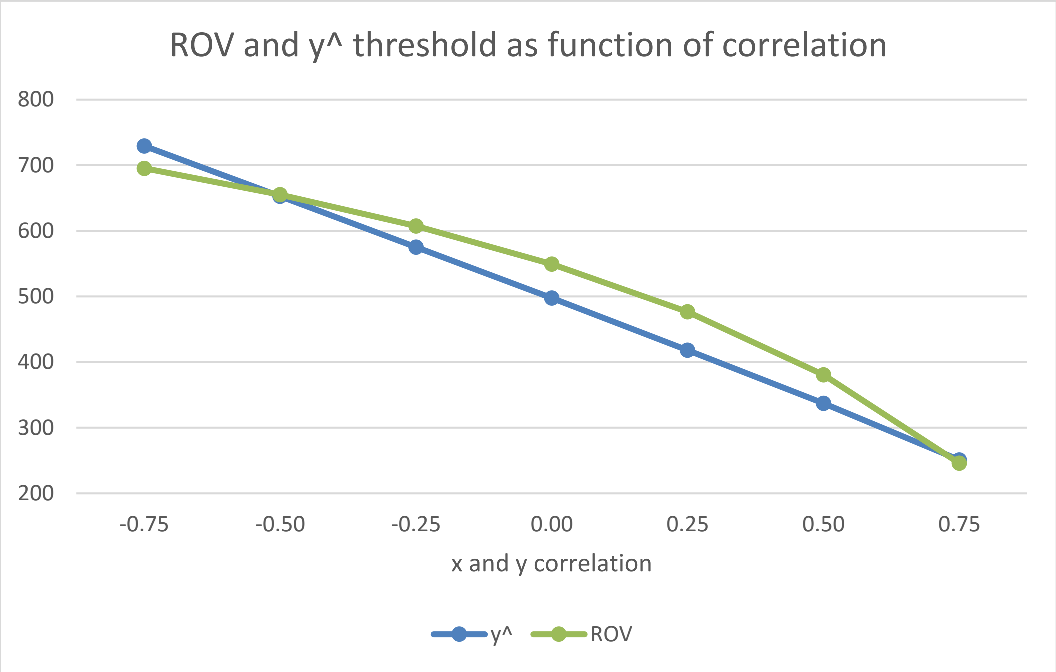
OO Switching output y Threshold Chi, sensitivity to changes in correlation.



Output y threshold and ROV both decline as x ,y correlation increases.

Figure 10

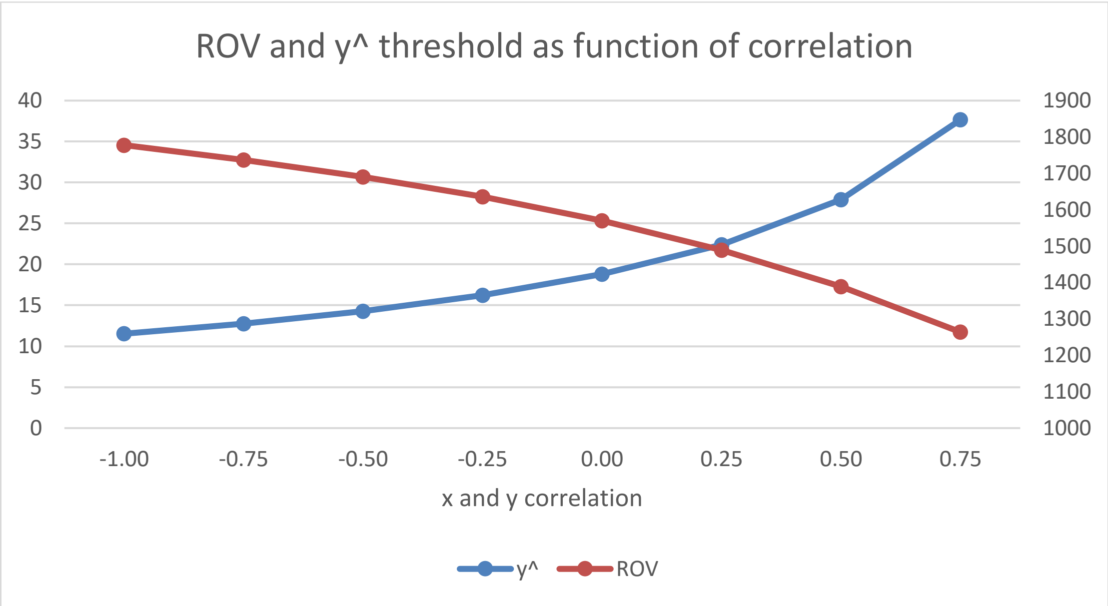
Output-Input ShutDown ROV and y threshold Chi



IO Shutdown ROV and input y threshold both decline as x , y correlation increases.

Figure 11

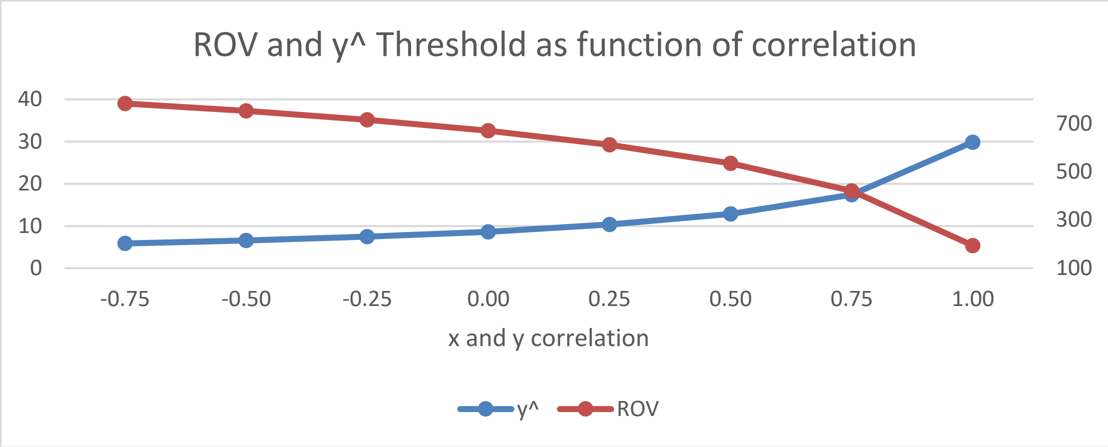
Output-Input StartUp ROV and y Threshold Chi



IO StartUp ROV coefficient A decreases but input y threshold increases as x, y correlation increases. With negative correlation, the spread between output and input is likely to be variable, so there is more optionality, thus value in waiting.

Figure12

Input-Input ROV and y Threshold Chi



II switching ROV decreases and y threshold increases as x,y correlation increases.

DISCUSSION

Illustrating the ROV, NPV, vegas and chi's when changing just one parameter value is perhaps more hypothetical than realistic. If there is positive correlation an increase y will generally be accompanied by an increase in x. Similarly, As noted by Støre et al. (2018), the covariance is not constant, if the correlation is constant while the x or y volatility changes as in Figures 9-12.

Why be concerned with the ROV and threshold sensitivities? Is the objective to achieve or maintain a high ROV (pending action)? Then. if x does not change, increasing y will promote that objective for all investment type switching opportunities. Allowing or encouraging high x volatility past a mid-point will promote that objective for all switching opportunities (if correlation is positive), as will allowing or encouraging negative correlation (but watch out for the effect on vegas). It is not clear how hedging either outputs or inputs, separately or together, thereby reducing volatility, is consistent with this objective.

Is an alternative objective to achieve or maintain a low y threshold (thereby motivating action)? Then if x does not change, allowing or maintaining medium x volatility near the inflexion point will promote that objective for OO and IO-SD switching opportunities (if correlation is positive), as will allowing or encouraging positive correlation. But the opposite holds for IO-SU and II switching opportunities. Suppose the manager seeks to justify switching (for instance from natural gas to natural gas liquids), or start-up investments in related drilling opportunities. Lower x or y volatility will lower the y threshold for a startup investment, but raise the output switching threshold, if correlation is positive. Government could also provide volatility lowering arrangements for some outputs and inputs.

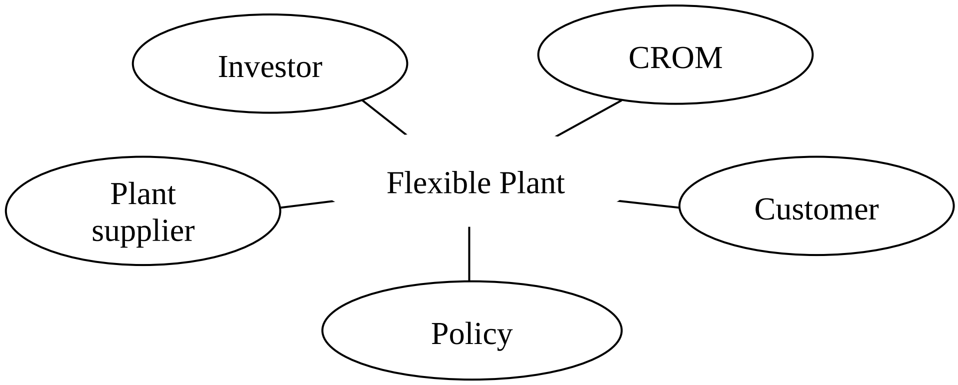
Correlation considerations are complex. If high ROV is the objective, then negative correlations are always desirable. But if the input is wages, what is the morality of not encouraging profit sharing? If low y thresholds are the objective, then high correlation promotes low thresholds for output switching, and abandonment. But low correlation promotes low thresholds for IO SU and II switching. The Appendix shows that high correlation results in high second order sensitivities Gamma x and Gamma y, implying that any hedging of correlation becomes highly sensitive to small correlation changes.

Finally, how might low or high correlations be achieved through hedging without affecting x or y volatility?

4 Policy and Strategy Implications

There are a number of stakeholders shown in Figure 13 whose best decisions should be based on these switching models.

Figure 13



Investors

As shown in Figures 1-4, the real option value of all of these switching opportunities is substantially greater than the present value of current production or NPV of the alternative, at the current assumed initial input and output price levels. Although some models and solutions are similar, each switching option is in a different context, where the y output must increase, or y as an input decrease, to justify a switching action. Note the focus of alert investors is on choosing the appropriate model and on forecasting input and output price volatilities and correlations. A myopic investment analyst using net present values will probably undervalue switching opportunities or flexible facilities. Analysts may not have access to plant operating or switching costs, or indeed knowledge of any flexibility inherent in existing facilities, due conceivably to inadequate accounting disclosures, not currently required by accounting standard setting committees. Of course, realistic analysts may doubt that the chief option managers of flexible facilities will be aware of the potential optionality, or indeed make switches at appropriate times, so the Marshallian values might reflect a realistic allowance for management shortfalls.

Chief Real Options Manager ("CROM")

The alert CROM is aware of input and output switching opportunities, the amount of switching costs, and periodically observes input and output prices, convenience yields (or proxies), updates expected volatilities and correlations, and so updates appropriate

Figures 1-4. Observed current spreads between input/output prices are compared to the updated triggers for switching, perhaps based on simple approximate linear rules over short or stable periods. Naturally part of the appropriate compensation for the CROM should be based on awareness of these opportunities, and performance in making actual input and output switches at appropriate times.

Originally, the CROM would have calculated the value of a flexible plant V_1, compared to an inflexible facility, which also indicates the warranted extra investment cost for facility flexibility. It would not be difficult to consider trade-offs for any deterministic lower efficiency due to the flexibility capacity.

Plant Suppliers

Originally, suppliers of facilities to the CROM would have calculated the value of a flexible plant V_1, compared to an inflexible facility, which also indicates the warranted extra investment price that could be charged for facility flexibility. With the illustrated parameter values, a hypothetical single switch plant is worth much more than an inflexible facility. In designing flexible facilities, it would not be difficult to consider trade-offs for any lower efficiency due to the flexibility capacity against increased building costs.

Customers

Output customers may be aware of the limitations, or capacities, of producers to switch to higher price products, opportunistically, or to alternative lower price inputs when appropriate. Input suppliers may become cautious with buyers, who switch sources optimally. Other customers might seek long-term agreements mitigating the shifts in output and input prices implied in using real option approaches for operating flexible facilities.

Policy Makers

Taxpayers beware. There will be national producers without flexible facilities, or not aware of needing to change output prices, and input sources, as the economic environment changes. Those producers priced out of the market will seek government barriers for other producers, or input/output subsidies as conditions change.

5 Applications[1]

Flexibility between outputs and inputs is particularly relevant in volatile commodity markets, or where free trade allows new entrants, cheaper inputs, or more valuable outputs. Think of the many applications for substitute outputs, substitute inputs, or alternative inputs and outputs. Dockendorf and Paxson (2013) examine further processed chemical products as essentially output alternatives. They note alternative uses of other types of facilities, such as multiuse sports or entertainment or educational facilities, transportation vehicles for passengers or cargo, rotating agricultural crops, and solar energy used for electricity or water desalination. Støre et al. (2018) applied the output switching model to producing natural gas rather than oil in a mature North Sea field. Adkins and Paxson (2011a) note there are numerous energy input switching opportunities, such as palm or rape oil in biodiesel production, gas-oil-hydro-coal in electricity generation, that are reciprocal energy input switching options. There are several examples of stochastic output and input prices, such as the "crack" spread for gasoline-heating oil as outputs for crude oil refineries, the "crush" spread for soya meal and soya oil as outputs for soya bean refineries, and ethanol the output for corn processing facilities.

[1] Readers are invited to view case studies in switching from drilling for natural gas to natural gas liquids for two Appalachian drillers, relevant for the current low natural gas price environment. Also, spreadsheet for the quasi-analytical and analytical solutions Figures A1-2. B1-2, C1-2 and D1-2 are available from the authors. Also available are listing of over 200+ plausible switching applications.

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APPENDIX

A. OUTPUT SWITCHING

Consider a flexible facility which can be used to produce one of two different outputs by switching once between operating modes. Assume the prices of the two outputs, x and y, are stochastic, possibly correlated and follow geometric Brownian motion (gBm):

$$dx = (\mu_x - \delta_x)x\,dt + \sigma_x\,x\,dz_x \quad \text{(A1)}$$

$$dy = (\mu_y - \delta_y)y\,dt + \sigma_y\,y\,dz_y \quad \text{(A1)}$$

with the notations: μ expected drift of the output price, δ convenience yield of the output, σ volatility of the output, ρ correlation between the two output prices `and dz Wiener process (stochastic element). The instantaneous cash flow in each operating mode is the respective commodity price of the output less unit operating cost, assuming production of one (equivalent) unit per annum, $(x - c_x)$ in operating mode '1' and $(y - c_y)$ in operating mode '2'. The operating costs c_x and c_y are per unit produced. A switching cost of SO is incurred when switching from operating mode '1' to '2'. Note one could start either with x or y, but it is logical that with the same operating costs, one would start with the highest output price. The appropriate discount rate is r for non-stochastic elements, such as constant operating costs. For convenience and simplicity, assume that the appropriate discount rate for the stochastic variables is δ, which is equal to r-μ.

Further assumptions are that the lifetime of the asset is infinite, the company is not restricted in the product mix choice because of selling commitments, and there is no competition. Moreover, the typical assumptions of real options theory apply, with interest rates, convenience yields, volatilities and correlation constant over time.

Quasi-analytical Solution for Output Switching

The asset value with opportunities to switch once between the two operating modes is given by the present value of perpetual cash flows in the current operating mode plus the option to switch to the alternative mode. Let V_1 be the asset value in operating mode '1', producing output x, and V_2 the asset value in operating mode '2', producing output y accordingly. The switching option depends on the two correlated stochastic variables x and y, and so do the asset value functions which are defined by the following partial differential equations (PDE):

$$\frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 V_1}{\partial y^2} + \rho\sigma_x\sigma_y xy \frac{\partial^2 V_1}{\partial x \partial y} + (r-\delta_x)x\frac{\partial V_1}{\partial x} + (r-\delta_y)y\frac{\partial V_1}{\partial y} - rV_1 + (x-c_x) = 0 \tag{A2}$$

$$\frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 V_2}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 V_2}{\partial y^2} + \rho\sigma_x\sigma_y xy \frac{\partial^2 V_2}{\partial x \partial y} + (r-\delta_x)x\frac{\partial V_2}{\partial x} + (r-\delta_y)y\frac{\partial V_2}{\partial y} - rV_2 + (y-c_y) = 0 \tag{A3}$$

(A3) assumes the initial operating state produces output x, with an option to switch once to y, while (A4) that the initial state produces output y, with an option to switch once to x. Two-factor problems which are linear homogeneous, i.e. $\mathrm{V}(\lambda \cdot \mathrm{x}; \lambda \cdot \mathrm{y}) = \lambda \cdot \mathrm{V}(\mathrm{x};\mathrm{y})$, can typically be solved analytically by substitution of variables, so that the PDE can be reduced to a one-factor differential equation. An example of this is the perpetual American exchange option in McDonald and Siegel (1986). With a constant switching cost and operating costs, the problem is no longer homogenous of degree one and the dimension reducing technique cannot be used.

Dockendorf and Paxson (2013) derive a quasi-analytical solution for a similar type of two-factor non-homogeneous problem. For two outputs, the PDEs are satisfied by the following general solution:

$$V_1(x,y) = A x^{\beta_2} y^{\beta_1} + \frac{x}{\delta_x} - \frac{c_x}{r} \tag{A4}$$

$$V_2(x,y) = Bx^{\beta_1} y^{\beta_2} + \frac{y}{\delta_y} - \frac{c_y}{r} \tag{A6}$$

where in A5 β_1 and β_2 satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_2(\beta_2-1) + \tfrac{1}{2}\sigma_y^2\beta_1(\beta_1-1) + \rho\sigma_x\sigma_y\beta_2\beta_1 + \beta_2(r-\delta_x) + \beta_1(r-\delta_y) - r = 0 \tag{A7}$$

Assuming $c_y \geq c_x$, and/or $x \geq y$, the American perpetual option to switch from x to y can be determined, so we will not consider the option value in (A6). The asset value V_1 is given by (AA4) with the characteristic root equation (A7), and V_2 is given by the RHS second and third terms of (A6), if B=0. Since the option to switch from x to y decreases with x and increases with y, β_2 must be negative and β_1 positive. A quasi-analytical solution is obtained by considering the value matching condition (A8):

$$A\hat{x}^{\beta_2}\,\hat{y}^{\beta_1}+\frac{\hat{x}}{\delta_x}-\frac{c_x}{r}=\frac{\hat{y}}{\delta_y}-\frac{c_y}{r}-S \tag{A8}$$

and the two smooth pasting conditions at the boundaries:

$$\beta_2\,A\hat{x}^{\beta_2-1}\,\hat{y}^{\beta_1}+\frac{1}{\delta_x}=0 \tag{A9}$$

$$\beta_1\,A\hat{x}^{\beta_2}\,\hat{y}^{\beta_1-1}-\frac{1}{\delta_y}=0 \tag{A10}$$

The characteristic root equation (A7) together with value matching condition (A8) and smooth pasting conditions (A9) and (A10) represent a system of 4 equations, while there are 5 unknowns, β_1, β_2, A, $\hat{x},\hat{y}$. A quasi-analytical solution is obtained by solving the 4 equations simultaneously, assuming $\hat{x}=x$, then deriving $\hat{y}$, thus for pairs of $\{\hat{x},\hat{y}\}$. The analytical solution is shown in the main text. Here are illustrative results for the single output switch model, assuming current operating costs are half of current gross revenue for each output. Figure A1 shows that the option coefficient A is positive, β_2 is negative and β_1 is positive, thereby fulfilling the requirements from the theoretical model. The solution satisfies the PDE (A3).

Figure A1

	A	B	C	D	E	F	G
1	**Continuous American Perpetual SINGLE SWITCH Option**						
2	ONE WAY SWITCH FROM OUTPUT X TO Y						
3	OUTPUT X	x	100				
4	OUTPUT Y	y	100				
5	Convenience yield of x	δx	0.04				
6	Convenience yield of y	δy	0.04				
7	Volatility of x	σx	0.40				
8	Volatility of y	σy	0.30				
9	Correlation x with y	ρ	0.50				
10	Risk-free interest rate	r	0.05				
11	Operating cost for x	cx	50				
12	Operating cost for y	cy	50				
13	Switching cost: x to y	S	50				
14	Intrinsic Switching NPV	Y-S	1450				
15	PV of revenues x	X	1500				
16	PV of revenues y	Y	1500				
17	Switching boundary x to y	x^	100				
18		ROV	**1028.979**				
19	Asset value in operating mode '1'	**V1(x,y)**	**2528.979**				
20	Asset value in operating mode '2'	**V2(x,y)**	1500.000				
21		A	9.894				
22	Switching boundary x to y	y^ (x^)	337.043				
23	Solution quadrant	β_2	-0.425				
24	Solution quadrant	β_1	1.434				
25		EQUATIONS					
26	Value matching	EQ 8	0.000				
27	Smooth pasting 1	EQ 9	0.000				
28	Smooth pasting 2	EQ 10	0.000				
29	Solution quadrant 1	EQ 7	0.000				
30	**Solver: C30=0, changing C21:C24.**	Sum	0.000				
31	SPREAD		**237.043**	C22-C4			
32	PDE	EQ 3	0.000				
33	ΔROV+x		20.622				
34	ΔROV+y		14.755				
35	ΓROV+x		0.062				
36	ΓROV+y		0.064				
37	ΓROV+x,y		-0.063				
38	Value matching 1 at y^		7,376.087				
39	Value matching 2 at y^- S		7,376.087				
40	EQ 8	(C21*C17^C23*C22^C24+C17/C5-C11/C10-C22/C6+C12/C10+C13)					
41	EQ 9	(C23*C21*C17^(C23-1)*C22^C24+1/C5)					
42	EQ 10	(C24*C21*C17^C23*C22^(C24-1)-1/C6)					
43	0.5*C7^2*C23*(C23-1)+0.5*C8^2*C24*(C24-1)+C9*C7*C8*C23*C24+C23*(C10-C5)+C24*(C10-C6)-C10						
44	0.5*(C7^2)*(C3^2)*C35+0.5*(C8^2)*(C4^2)*C36+C9*C7*C8*C3*C4*C37+(C10-C5)*C3*C33+(C10-C6)*C4*C34-C10*C19+(C3-C11)						
45	ΔROV+x	C23*C21*(C3^(C23-1))*(C4^C24)+1/C5					
46	ΔROV+y	C24*C21*(C3^C23)*(C4^(C24-1))					
47	ΓROV+x	C23*(C23-1)*C21*(C3^(C23-2))*(C4^C24)					
48	ΓROV+y	C24*(C24-1)*C21*(C3^C23)*(C4^(C24-2))					
49	ΓROV+x,y	C23*C24*C21*(C3^(C23-1))*(C4^(C24-1))					
50	Value matching 1 at y^	C21*C3^C23*C22^C24+C3/C5-C11/C10					
51	Value matching 2 at y^- S	C22/C6-C12/C10-C13					

The asset values are given in the modes, V_1 operating with x, and V_2, operating with y (without the opportunity to switch back to x), and the level of y is indicated when it is optimal to switch from x to y, when $\hat{x} = x = 100$. In this example, with x and y having the same initial values and the same convenience yields, the asset value excluding the switching option value is identical in both operating modes when the operating cost is the same, cells C15:C16. Higher operating costs reduce the asset value. When operating costs are 50, the asset value V_1 with a continuous switching opportunity is valued at 2529 if the incumbent is x=100 with a volatility of 40% according to the quasi-analytical solution. The switching option value is the difference between the asset value and the value with no switching option, 2529-1500=1029. The option to switch between the two operating modes once adds about 69% to the inflexible asset value. Switching to output y is justified if y increases to 337% higher than the current output y. The spread between y and $\hat{y}$ is due to switching costs and stochastic elements, and increases with high volatilities and low correlation, following real options theory. It should be noted that changing x also changes the switching boundary $\hat{y}$.

Figure A2

	A	B	C	D	E	F	G	H	I
1		Simplified Støre et al. 2018 Analytical Solution Output Switching							
2		INPUTS							
3		x	100						
4		y	100						
5		δx	0.04						
6		δy	0.04						
7		σx	0.40						
8		σy	0.30						
9		ρ	0.50						
10		r	0.05						
11		cx	50						
12		cy	50						
13		S	50						
14		Y-S	1450						
15		X	1500	(C3/C5-C11/C10)					
16		Y	1500	(C4/C6-C12/C10)					
17		x^	100	C3					
18		OUTPUTS							
19		C(x^)	1.02000	1+(C5)/(C3)*(C13)					
20		f(x^)	0.132	(C7^2)-(2*(C10-C5))-(2*C9*C7*C8)+(C19*(2*(C10-C6)+(C8^2)))					
21		g(x^)	0.131	(C7^2)+((C8^2)*(C19^2))-(2*C9*C7*C8*C19)					
22		β_2(x^)	-0.425	(C20/(2*C21))-SQRT(((C20/(2*C21))^2)+(2*((C6)/C21)))					
23		β_1(x^)	1.434	1-C19*C22					
24		y^	**337.043**	**(-C23*C6*C17)/(C22*C5)**					
25		A	9.894	(-1/(C22*C5*(C17^(C22-1))*(C24^C23)))					
26		ROV	**1028.979**	C25*(C3^C22)*(C4^C23)					
27		VALUE	**2528.979**	C26+C3/(C5)-C11/C10					
28		PDE	0.000						
29		ΔROV+x	20.622	C22*C25*(C3^(C22-1))*(C4^C23)+1/(C5)					
30		ΔROV+y	14.755	C23*C25*(C3^C22)*(C4^(C23-1))					
31		ΓROV+x	0.062	C22*(C22-1)*C25*(C3^(C22-2))*(C4^C23)					
32		ΓROV+y	0.064	C23*(C23-1)*C25*(C3^C22)*(C4^(C23-2))					
33		ΓROV+x,y	-0.063	C22*C23*C25*(C3^(C22-1))*(C4^(C23-1))					
34	Value matching 1 at y^		7,376.087	C25*(C17^C22)*(C24^C23)+C17/C5-C11/C10					
35	Value matching 2 at y^- :		7,376.087	C24/C6-C12/C10-C13					
36	PDE	0.5*(C7^2)*(C3^2)*C31+0.5*(C8^2)*(C4^2)*C32+C9*C7*C8*C3*C4*C33+(C10-C5)*C3*C29+(C10-C6)*C4*C30-C10*C27+(C3-C11)							
37		ROV	1028.979	(-C3/(C22*C5))*((C4/C24)^C23)					

Figure A2 shows that the analytical solution arrives at exactly the same results. One advantage of an analytical solution is that it is very easy to copy the columns in Excel, changing one parameter value.

The ROV+ partial derivatives are:

$$\Delta ROV+,x = \beta_2 A x^{\beta_2 -1} y^{\beta_1} + \frac{1}{\delta_x} \tag{A11}$$

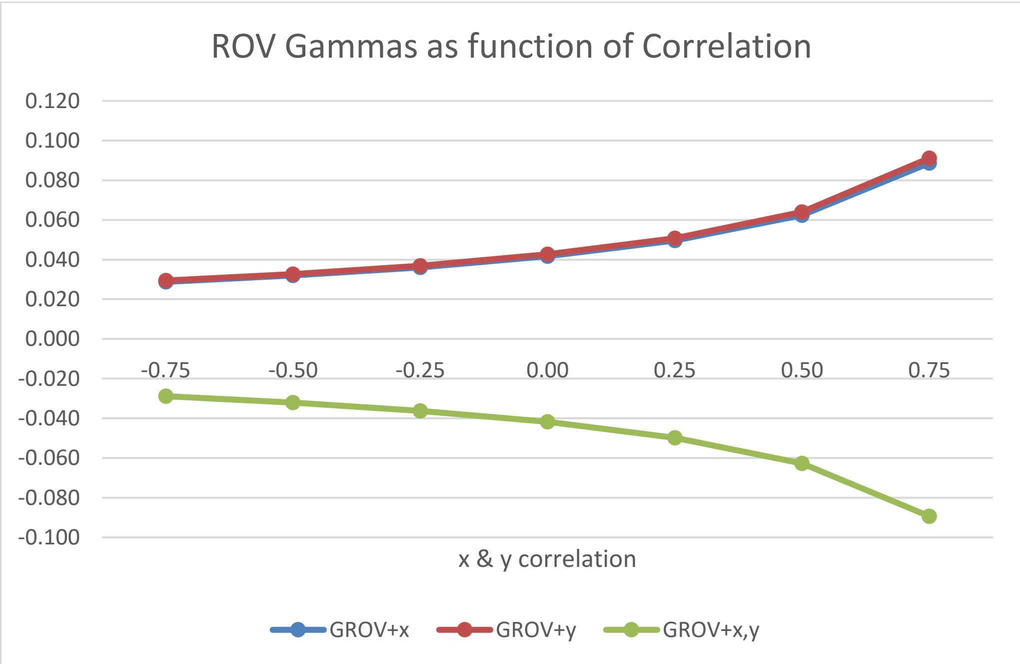
$$\Delta ROV+,y = \beta_1 A x^{\beta_2} y^{\beta_1 -1} \tag{A12}$$

$$\Gamma ROV+,x = \beta_2(\beta_2 -1) A x^{\beta_2 -1} y^{\beta_1} \tag{A13}$$

$$\Gamma ROV+,y = \beta_1(\beta_1 -1) A x^{\beta_2} y^{\beta_1 -2} \tag{A14}$$

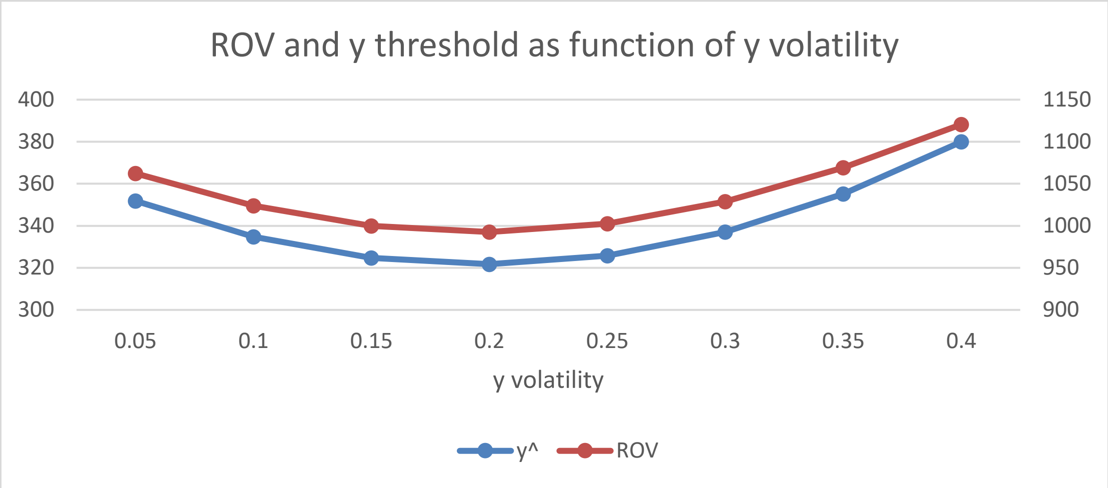
$$\Gamma ROV+,x,y = \beta_2 \beta_1 A x^{\beta_2 -1} y^{\beta_1 -1} \tag{A15}$$

Figure A3



ROV x and y Gammas (A13) (A14) increase with correlation, but the cross-gammas (A15) decrease.

Figure A4



Both the ROV and y threshold vegas first decrease, then increase with increases in y volatility. Note that Figure A4 is similar to Figure 5.

B. INPUT-OUTPUT SWITCHING: SHUTDOWN

The asset value with an opportunity to switch once from an operating mode to an abandoned state with an abandonment cost (D) but no salvage value (when both inputs and outputs are stochastic) is given by the present value of perpetual cash flows in the current operating mode plus the option to abandon. Let V_1 be the asset value in operating mode '1', producing output x at input cost y. Following Adkins and Paxson (2012) the switching option depends on the two possibly correlated stochastic variables x and y, and so does the asset value function which is defined by the following PDE:

(B1)

$$\frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 V_1}{\partial y^2} + \rho\sigma_x\sigma_y xy \frac{\partial^2 V_1}{\partial x \partial y} + (r-\delta_x)x\frac{\partial V_1}{\partial x} + (r-\delta_y)y\frac{\partial V_1}{\partial y} - rV_1 + (x-y) = 0$$

This one-way switch constitutes an abandonment option, where the switching cost is the abandonment cost.

$$V_1(x,y) = Ax^{\beta_2} y^{\beta_1} + \frac{x}{\delta_x} - \frac{y}{\delta_y} \quad \text{(B2)}$$

where β_1 (positive root) and β_2 satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_2(\beta_2-1) + \tfrac{1}{2}\sigma_y^2\beta_1(\beta_1-1) + \rho\sigma_x\sigma_y\beta_2\beta_1 + \beta_2(r-\delta_x) + \beta_1(r-\delta_y) - r = 0 \quad \text{(B3)}$$

since the option to switch from operating to abandonment decreases with x and increases with y. Assuming initially $y < \hat{y}$, the asset value V is given by (B2) with the characteristic root equation (B3). Applying the standard procedure, a quasi-analytical solution is obtained.

$$A\hat{x}^{\beta_2}\,\hat{y}^{\beta_1} + \frac{\hat{x}}{\delta_x} - \frac{\hat{y}}{\delta_y} + D = 0 \tag{B4}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_2\,A\hat{x}^{\beta_2-1}\,\hat{y}^{\beta_1} + \frac{1}{\delta_x} = 0 \tag{B5}$$

$$\beta_1\,A\hat{x}^{\beta_2}\,\hat{y}^{\beta_1-1} - \frac{1}{\delta_y} = 0 \tag{B6}$$

The characteristic root equation (B3) together with value matching condition (B4) and smooth pasting conditions (B5) and (B6) represent a system of 4 equations, while there are 5 unknowns, β_1, β_2, A, $\hat{x}, \hat{y}$. The problem is solved assuming $\hat{x} = x$, then deriving $\hat{y}$.

The analytical solution is similar to that given in the text.

The value of an operating state which entails the opportunity to abandon is the RHS of (B2), where the first part is the value of the real option ROV to abandon, and the second part is the current perpetual value of producing with output x and input y, which together are referred to as $ROV+ = V_1(x, y)$.

The ROV+ partial derivatives are:

$$\Delta ROV+, x = \beta_2 A x^{\beta_2-1} y^{\beta_1} + \frac{1}{\delta_x} \tag{B7}$$

$$\Delta ROV+, y = \beta_1 A x^{\beta_2} y^{\beta_1-1} - \frac{1}{\delta_y} \tag{B8}$$

$$\Gamma ROV+, x = \beta_2(\beta_2 - 1) A x^{\beta_2-1} y^{\beta_1} \tag{B9}$$

$$\Gamma ROV+, y = \beta_1(\beta_1 - 1) A x^{\beta_2} y^{\beta_1-2} \tag{B10}$$

$$\Gamma ROV+, x, y = \beta_2\beta_1 A x^{\beta_2-1} y^{\beta_1-1} \tag{B11}$$

Figure B1

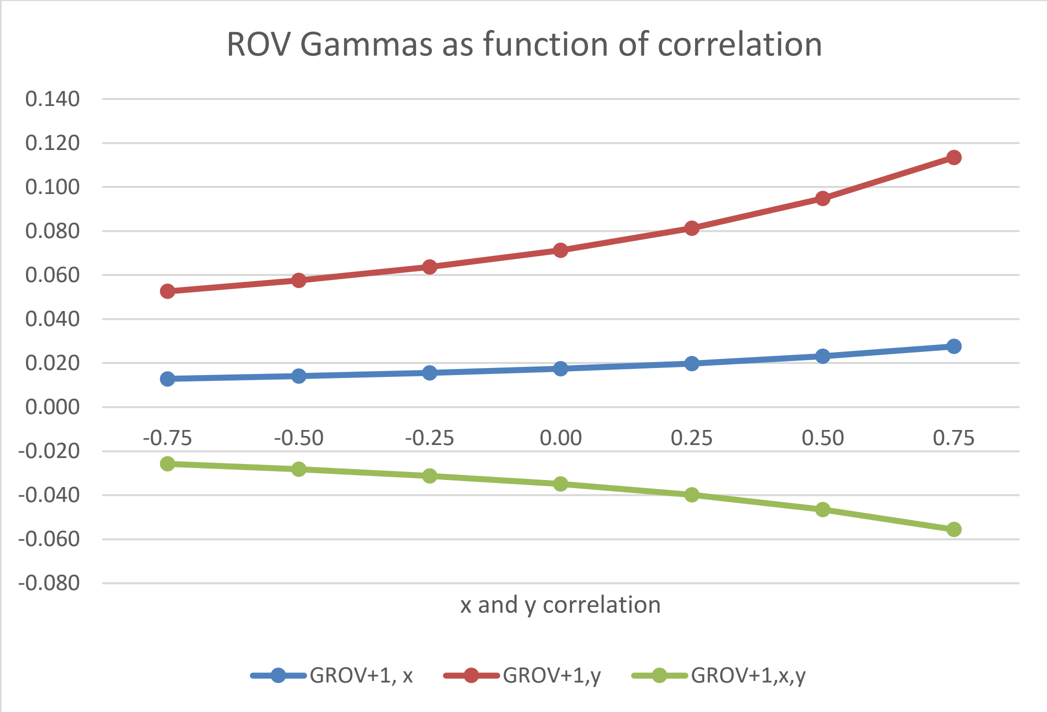
	A	B	C	D	E	F	G
1	**SINGLE American Perpetual IN-OUT SWITCH Option Non-homogeneous**						
2		INPUT		Shut Down			
3	PRICE	x	100				
4	COST	y	50				
5	Convenience yield of x	δ_x	0.04				
6	Convenience yield of y	δ_y	0.04				
7	Volatility of x	σ_x	0.40				
8	Volatility of y	σ_y	0.30				
9	Correlation x with y	ρ	0.50				
10	Risk-free interest rate	r	0.05				
11	Switching cost from x to y	D	50				
12	Switching boundary OP to SHUT	x	100				
13		OPTION	**380.84**				
14	Asset value in operating mode	**V_1(x,y)**	**1630.84**				
15		A	9.89				
16	Switching boundary OP to SHUT	y^(x)	**337.043**				
17	Solution quadrant	β_2	-0.425	*must be negative*			
18	Solution quadrant	β_1	1.434	*must be positive*			
19	OPERATING		**1250.00**				
20	Value matching 1		0.000				
21	Smooth pasting 1A		0.000				
22	Smooth pasting 1B		0.000				
23	Solution quadrant 1		0.000				
24		Sum	0.000	**Solver**			
25		SOLVER: SET C24=0, CHANGING C15:C18					
26	SPREAD		**287.04**				
27	V_1(x,y)	(C15*C3^C17*C4^C18+C3/C5-C4/C6)					
28	Value matching 1	(C15*C12^C17*C16^C18+C12/C5-C16/C6+C11)					
29	Smooth pasting 1A	(C17*C15*(C12^(C17-1))*C16^C18+1/C5)					
30	Smooth pasting 1B	(C18*C15*C12^C17*(C16^(C18-1))-1/C6)					
31	Solution quadrant 1	0.5*C7^2*C17*(C17-1)+0.5*C8^2*C18*(C18-1)+C9*C7*C8*C17*C18+C17*(C10-C5)+C18*(C10-C6)-C10					
32	PDE		0.0000				
33	ΔROV+1,x		23.3797				
34	ΔROV+1,y		-14.0778				
35	ΓROV+1, x		0.0231				
36	ΓROV+1,y		0.0948				
37	ΓROV+1,x,y		-0.0465				
38	0.5*(C7^2)*(C3^2)*C35+0.5*(C8^2)*(C4^2)*C36+C9*C7*C8*C3*C4*C37+(C10-C5)*C3*C33+(C10-C6)*C4*C34-C10*C14+(C3-C4)						
39	ΔROV+1,x		C15*C17*(C3^(C17-1))*(C4^C18)+1/C5				
40	ΔROV+1,y		C15*C18*(C3^(C17))*(C4^(C18-1))-1/C6				
41	ΓROV+1, x		C15*C17*(C17-1)*(C3^(C17-2))*(C4^C18)				
42	ΓROV+1,y		C15*C18*(C18-1)*(C3^(C17))*(C4^(C18-2))				
43	ΓROV+1,x,y		C15*C17*C18*(C3^(C17-1))*(C4^(C18-1))				

The analytical solution, adapting Støre et al. (2018), as simplified in Adkins and Paxson (2018), gives exactly the same results.

Figure B2

	A	B	C
1	Simplified Støre et al. 2018 Analytical Solution IN-OUT Switching		
2	INPUTS	Shut Down	
3	x	100	
4	y	50	
5	δx	0.04	
6	δy	0.04	
7	σx	0.40	
8	σy	0.30	
9	ρ	0.50	
10	r	0.05	
11	cx		
12	cy		
13	D	50.000	
14			
15	X	1250	(B3/B5-B11/B10)
16			
17	x^	100	B3
18	OUTPUTS		
19	C(x^)	1.020	1+(B5)/(B3)*(B13)
20	f(x^)	0.132	(B7^2)-(2*(B10-B5))-(2*B9*B7*B8)+(B19*(2*(B10-B6)+(B8^2)))
21	g(x^)	0.131	(B7^2)+((B8^2)*(B19^2))-(2*B9*B7*B8*B19)
22	β_2(x^)	-0.425	(B20/(2*B21))-SQRT(((B20/(2*B21))^2)+(2*((B6)/B21)))
23	β_1(x^)	1.434	1-B19*B22
24	y^	**337.043**	**(-B23*B6*B17)/(B22*B5)**
25	A	9.894	(-1/(B22*B5*(B17^(B22-1))*(B24^B23)))
26	ROV	**380.838**	**IF(B4<B24,B25*(B3^B22)*(B4^B23),B27)**
27	-NPV	**0.000**	**MAX(-(B3/B5-B4/B6-B13),0)**
28	PDE	0.000	
29	ΔROV+1,x	23.380	B22*B25*(B3^(B22-1))*(B4^B23)+1/(B5)
30	ΔROV+1,y	-14.078	B23*B25*(B3^B22)*(B4^(B23-1))-1/B6
31	ΓROV+1, x	0.023	B22*(B22-1)*B25*(B3^(B22-2))*(B4^B23)
32	ΓROV+1,y	0.095	B23*(B23-1)*B25*(B3^B22)*(B4^(B23-2))
33	ΓROV+1,x,y	-0.046	B22*B23*B25*(B3^(B22-1))*(B4^(B23-1))
34	VALUE	**1630.838**	**B26+B3/B5-B4/B6**
35	ROV	380.838	(-B3/(B22*B5))*((B4/B24)^B23)

Figure B3



The IN-OUT shut-down ROV gammas are positive and increasing for x (and slightly for y) as the correlation increases, and negative and decreasing cross-gamma for x,y.

C. INPUT-OUTPUT SWITCHING: STARTUP

The asset value with an opportunity to switch once from an idle state to an operating mode (when both inputs and outputs are stochastic) is given by equating the value of the option to invest with the present value of perpetual cash flows in the operating mode less the investment cost. Obviously, this is a basic real option investment model but considering the volatility of the two stochastic factors separately along with the correlation. Let V_1 be the asset value in the idle mode '1'representing the opportunity to invest in an operating mode with output x at input cost y, assuming unit quantity of production. The switching start-up option depends on the two possibly correlated stochastic variables x and y, and so does the asset value function V_1 which is defined by the following PDE:

(C1)

$$\frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 V_1}{\partial y^2} + \rho\sigma_x\sigma_y xy \frac{\partial^2 V_1}{\partial x \partial y} + \left(r-\delta_x\right)x\frac{\partial V_1}{\partial x} + \left(r-\delta_y\right)y\frac{\partial V_1}{\partial y} - rV_1 = 0$$

The solution for (C1) is the fundamental investment option, now with two factors,

$$V_1(x, y) = A x^{\beta_1} y^{\beta_2} \tag{C2}$$

where β_1 (positive root) and β_2 satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_1(\beta_1 - 1) + \tfrac{1}{2}\sigma_y^2\beta_2(\beta_2 - 1) + \rho\sigma_x\sigma_y\beta_1\beta_2 + \beta_1(r - \delta_x) + \beta_2(r - \delta_y) - r = 0 \tag{C3}$$

since the value of the option to switch from idle to operating increases with x and decreases with y. Assuming K= investment cost, the value matching condition is:

$$A\hat{x}^{\beta_1}\hat{y}^{\beta_2} - \frac{\hat{x}}{\delta_x} + \frac{\hat{y}}{\delta_y} + K = 0 \tag{C4}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_1 A\hat{x}^{\beta_1 - 1}\hat{y}^{\beta_2} - \frac{1}{\delta_x} = 0 \tag{C5}$$

$$\beta_2 A\hat{x}^{\beta_1}\hat{y}^{\beta_2 - 1} + \frac{1}{\delta_y} = 0 \tag{C6}$$

The characteristic root equation (C3) together with value matching condition (C4) and smooth pasting conditions (C5) (C6) represent a system of 4 equations, while there are 5 unknowns, β_1, β_2, A, $\hat{x}, \hat{y}$. The problem is solved assuming $\hat{x} = x$, then deriving $\hat{y}$.

Following Støre et al. (2018), as simplified in Adkins and Paxson (2018), from (C5) and (C6):

$$\hat{y} = -\frac{\beta_2\delta_y\hat{x}}{\beta_1\delta_x} \tag{C7}$$

$$A = \frac{1}{\beta_1\delta_x\hat{x}^{\beta_1 - 1}\hat{y}^{\beta_2}} \tag{C8}$$

Assuming that the production output is per unit, let

$$C(\hat{x}) = 1 - \frac{\delta_x}{\hat{x}}K \tag{C9}$$

Adapting Støre et al. (2018), a solution for $\beta_1(\hat{x})$ is given by:

$$\beta_1(\hat{x}) = \frac{f(\hat{x})}{2g(\hat{x})} + \sqrt{\left(\frac{f(\hat{x})}{2g(\hat{x})}\right)^2 + \frac{2\delta_y}{g(\hat{x})}} \tag{C10}$$

where $f(\hat{x}) = \sigma_x^2 - 2(r - \delta_x) - 2\rho\sigma_x\sigma_y + C(\hat{x})(2(r - \delta_y) + \sigma_y^2)$ (C11)

and $g(\hat{x}) = \sigma_x^2 + \sigma_y^2 C(\hat{x})^2 - 2\rho\sigma_x\sigma_y C(\hat{x})$ (C12)

$$\beta_2 = 1 - \beta_1(\hat{x})C(\hat{x})\,. \tag{C13}$$

Substituting β_1 and β2 into (C7) and (C8) yields the analytical solutions for $\hat{y}(\hat{x})$ and the ROV.

$$ROV = Ax^{\beta_1}y^{\beta_2} = \frac{x^{\beta_1}y^{\beta_2}}{\beta_1\delta_x\hat{x}^{\beta_1-1}\hat{y}^{\beta_2}} = \frac{\hat{x}}{\beta_1\delta_x}\left(\frac{y}{\hat{y}}\right)^{\beta_2} \tag{C14}$$

Figure C1

	A	B	C	D	E	F	G
1	**SINGLE American Perpetual IN-OUT START UP Option**						
2		INPUT	***From Idle to Operating***				
3	PRICE	x	100				
4	COST	y	50				
5	Convenience yield of x	δ_x	0.04				
6	Convenience yield of y	δ_y	0.04				
7	Volatility of x	σ_x	0.40				
8	Volatility of y	σ_y	0.30				
9	Correlation x with y	ρ	0.50				
10	Risk-free interest rate	r	0.05				
11	Switching cost from x to y	K	50				
12	Switching boundary IDLE TO OP	x	100				
13		$V_2(x,y)$	**1250.000**				
14	Asset value in idle mode	$\mathbf{V_1(x,y)}$	**1388.939**				
15		A	9.239				
16	Switching boundary IDLE TO OP	y^(x)	27.927				
17	Solution quadrant	β_2	-0.399	*must be negative*			
18	Solution quadrant	β_1	1.427	*must be positive*			
19		EQUATIONS					
20	Value matching		0.000				
21	Smooth pasting 1		0.000				
22	Smooth pasting 2		0.000				
23	Solution quadrant		0.000				
24		Sum	0.000	**Solver**			
25		SOLVER: SET C24=0, CHANGING C15:C18					
26	SPREAD		**-22.07**				
27	$V_1(x,y)$	(C15*C3^C18*C4^C17)					
28	Value matching	(C15*C12^C18*C16^C17-C12/C5+C16/C6+C11)					
29	Smooth pasting 1	C18*C15*(C12^(C18-1))*C16^C17-1/C5					
30	Smooth pasting 2	C17*C15*C12^C18*(C16^(C17-1))+1/C6					
31	Solution quadrant	0.5*C7^2*C18*(C18-1)+0.5*C8^2*C17*(C17-1)+C9*C7*C8*C17*C18+C18*(C10-C5)+C17*(C10-C6)-C1					
32	PDE		0.0000				
33	ΔROV+1,x		19.8212				
34	ΔROV+1,y		-11.0708				
35	ΓROV+1, x		0.0847				
36	ΓROV+1,y		0.3097				
37	ΓROV+1,x,y		-0.1580				
38	0.5*(C7^2)*(C3^2)*C35+0.5*(C8^2)*(C4^2)*C36+C9*C7*C8*C3*C4*C37+(C10-C5)*C3*C33+(C10-C6)*C4*C34-C10*C14						
39	ΔROV+1,x		C15*C18*(C3^(C18-1))*(C4^C17)				
40	ΔROV+1,y		C15*C17*(C3^(C18))*(C4^(C17-1))				
41	ΓROV+1, x		C15*C18*(C18-1)*(C3^(C18-2))*(C4^C17)				
42	ΓROV+1,y		C15*C17*(C17-1)*(C3^(C18))*(C4^(C17-2))				
43	ΓROV+1,x,y		C15*C17*C18*(C3^(C18-1))*(C4^(C17-1))				

The analytical solution, adapting Støre et al. (2018), as simplified in Adkins and Paxson (2018), gives exactly the same results.

Figure C2

	A	B	C
1	Simplified Støre et al. 2018 Analytical Solution IN-OUT StartUp Switching		
2	INPUTS		
3	x	100	
4	y	50	
5	δx	0.04	
6	δy	0.04	
7	σx	0.4	
8	σy	0.3	
9	ρ	0.5	
10	r	0.05	
11			
12			
13	K	50	
14			
15	X	1250.000	B3/B5-B4/B6
16			
17	x^	100	B3
18	OUTPUTS		
19	C(x^)	0.980	1-(B5)/(B3)*(B13)
20	f(x^)	0.128	(B7^2)-(2*(B10-B5))-(2*B9*B7*B8)+(B19*(2*(B10-B6)+(B8^2)))
21	g(x^)	0.129	(B7^2)+((B8^2)*(B19^2))-(2*B9*B7*B8*B19)
22	β_1(x^)	1.4271	(B20/(2*B21))+SQRT(((B20/(2*B21))^2)+(2*((B6)/B21)))
23	β_2(x^)	-0.3985	1-B19*B22
24	y^	**27.927**	**(-B23*B6*B17)/(B22*B5)**
25	A	9.239	(1/(B22*B5*(B17^(B22-1))*(B24^B23)))
26	ROV	**1388.939**	**B25*(B3^B22)*(B4^B23)**
27	VALUE	**1388.939**	**B25*(B3^B22)*(B4^B23)**
28	PDE	0.0000	
29	ΔROV+1,x	19.8212	B22*B25*(B3^(B22-1))*(B4^B23)
30	ΔROV+1,y	-11.0708	B23*B25*(B3^B22)*(B4^(B23-1))
31	ΓROV+1, x	0.0847	B22*(B22-1)*B25*(B3^(B22-2))*(B4^B23)
32	ΓROV+1,y	0.3097	B23*(B23-1)*B25*(B3^B22)*(B4^(B23-2))
33	ΓROV+1,x,y	-0.1580	B22*B23*B25*(B3^(B22-1))*(B4^(B23-1))
34	0.5*(B7^2)*(B3^2)*B31+0.5*(B8^2)*(B4^2)*B32+B9*B7*B8*B3*B4*B33+(B10-B5)*B3*B29+(B10-B6)*B4*B30-B10*B27		
35	ΔROV+1,x PDE	67.7212	0.5*(B7^2)*(B3^2)*B31
36	ΔROV+1,y PDE	34.8365	0.5*(B8^2)*(B4^2)*B32
37	ΓROV+1, x PDE	-47.3966	B9*B7*B8*B3*B4*B33
38	ΓROV+1,y PDE	19.8212	(B10-B5)*B3*B29
39	ΓROV+1,x,y PDE	-5.5354	(B10-B6)*B4*B30
40	r ROV	-69.4470	-B10*B27
41	PDE	0.0000	SUM(B35:B40)
42	ROV	1388.939	(B3/(B22*B5))*((B4/B24)^B23)

The ROV partial derivatives are:

$$\Delta ROV, x = \beta_1 A x^{\beta_1 - 1} y^{\beta_2} \tag{C15}$$

$$\Delta ROV, y = \beta_2 A x^{\beta_1} y^{\beta_2 - 1} \tag{C16}$$

$$\Gamma ROV, x = \beta_1(\beta_1 - 1) A x^{\beta_1 - 1} y^{\beta_2} \tag{C17}$$

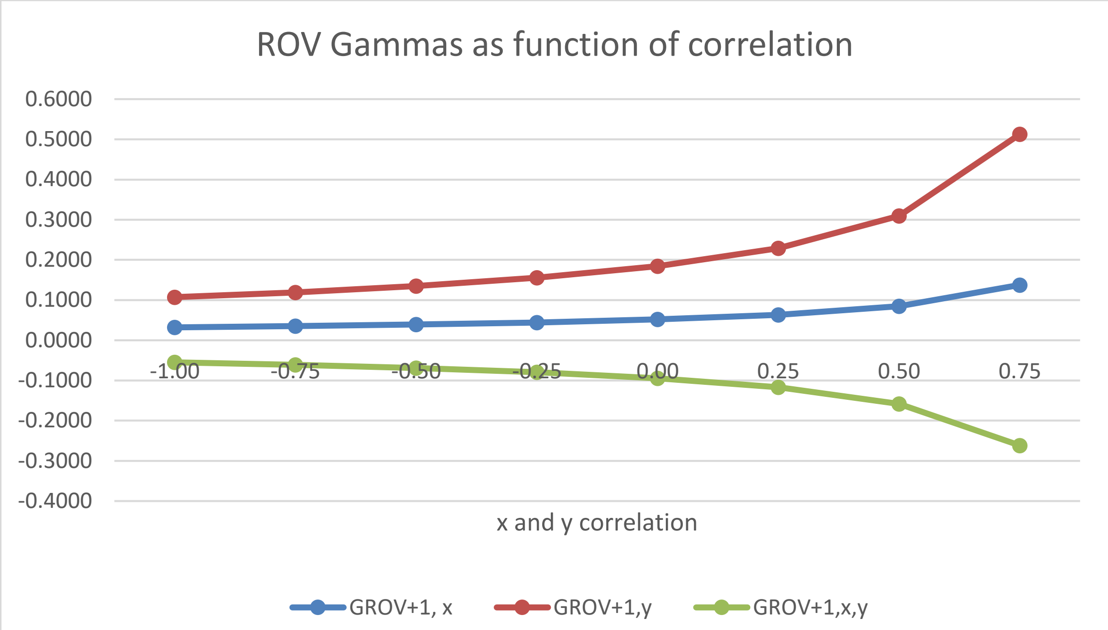
$$\Gamma ROV, y = \beta_2(\beta_2 - 1) A x^{\beta_1} y^{\beta_2 - 2} \tag{C18}$$

$$\Gamma ROV, x, y = \beta_1 \beta_2 A x^{\beta_1 - 1} y^{\beta_2 - 1} \tag{C19}$$

C32 in Figure C1 and B28 in Figure C2 show that the PDE (C1) is solved, based on the partial derivatives (C15)-(C19).

Similar to some other two stochastic factor vegas (sensitivity of option value or threshold to changes in expected output x volatility), the threshold is not monotonic, first increasing with volatility, then decreasing, in contrast to many other real option models, as shown in Figure 7.

Figure C3



ROV gammas (rate of delta change) are somewhat different for x and y as correlation increases. ΓROV y (x slightly) increases with correlation. The cross-gamma ΓROVx,y decreases with increases in correlation. While the gammas are small, when multiplied by $\rho\sigma_x\sigma_y xy, x(r-\delta_x), y(r-\delta_y)$ respectively from (C2), the dimension is increased in the PDE as shown in cells B37:B39 in Figure C2. These create problems for practical hedging of x and/or y, or x and/or y volatility.

D INPUT SWITCHING WITH A CONSTANT SWITCHING COST

Consider a flexible facility which can use one of two different inputs by switching once between operating modes. Assume the prices of the two inputs x and y, are stochastic, possibly correlated and follow gBm:

$$dx = (\mu_x - \delta_x)x\,dt + \sigma_x\,x\,dz_x \tag{D1}$$

$$dy = (\mu_y - \delta_y)y\,dt + \sigma_y\,y\,dz_y \tag{D2}$$

with the notations: μ is the expected drift of the input price, δ is the convenience yield of the input, σ the volatility of the input, dz Wiener process (stochastic element), and ρ is the correlation between the two input prices: $dz_x\,dz_y$ / dt.

The instantaneous cash flow in each operating mode is the unit output price less the respective price of the input, assuming production of one (equivalent) unit per annum, (p-x) in operating mode '1' and (p-y) in operating mode '2'. A switching cost of SI is incurred when switching from operating mode '1' to '2', assuming that output prices are equal but $x \leq y$, so currently the present value of operating in mode 1 with x is at least greater than or equal to mode 2. The appropriate discount rate is r for non- stochastic elements, such as constant output prices. For convenience and simplicity, assume that the appropriate discount rate for stochastic variables is δ which is equal to r-μ.

Further assumptions are that the output price is constant, the lifetime of the asset is infinite, and the company is not restricted in the input mix choice because of quality requirements or operating efficiency. Moreover, the typical assumptions of real options theory apply, with interest rates, convenience yields, volatilities and correlation constant over time.

Quasi-analytical Solution for Input Switching

The asset value with an opportunity to switch once between the two operating modes is given by the present value of perpetual cash flows in the current operating mode plus the option to switch to the alternative mode. Let V_1 be the asset value in operating mode '1', using input x. The switching option depends on the two correlated stochastic variables x and y, and so does the asset value function which is defined by the following PDE:

$$\frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 V_1}{\partial y^2} + \rho\sigma_x\sigma_y xy \frac{\partial^2 V_1}{\partial x \partial y} + \left(r - \delta_x\right) x \frac{\partial V_1}{\partial x} + \left(r - \delta_y\right) y \frac{\partial V_1}{\partial y} - rV_1 + \left(p_x - x\right) = 0 \tag{D3}$$

Adkins and Paxson (2011a) derive a quasi-analytical solution for a similar type of two-factor non-homogeneous problem. For two inputs, the PDE is satisfied by the following general solution:

$$V_1\left(x, y\right) = A x^{\beta_1} y^{\beta_2} + \frac{p_x}{r} - \frac{x}{\delta_x} \tag{D4}$$

Assuming one starts with input[2] x, the American perpetual option to switch from x to y can be determined. A quasi-analytical solution is obtained from the value matching condition, where the option to switch from input x to y increases with x so β_1>1` and decreases with y so β_2<0:

$$A\hat{x}^{\beta_1}\,\hat{y}^{\beta_2}+\frac{p_x}{r}-\frac{\hat{x}}{\delta_x}=\frac{p_y}{r}-\frac{\hat{y}}{\delta_y}-S \tag{D5}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_1\,A\hat{x}^{\beta_1-1}\,\hat{y}^{\beta_2}-\frac{1}{\delta_x}=0 \tag{D6}$$

$$\beta_2\,A\hat{x}^{\beta_1}\,\hat{y}^{\beta_2-1}+\frac{1}{\delta_y}=0 \tag{D7}$$

where β_1 and β_2 satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_1\left(\beta_1-1\right)+\tfrac{1}{2}\sigma_y^2\beta_2\left(\beta_2-1\right)+\rho\sigma_x\sigma_y\beta_1\beta_2+\beta_1\left(r-\delta_x\right)+\beta_2\left(r-\delta_y\right)-r=0 \tag{D8}$$

The characteristic root equation (D8) together with value matching condition (D5) and smooth pasting conditions (D6) and (D7) represents a system of 4 equations, while there are 5 unknowns, β_1, β_2, A, $\hat{x},\hat{y}$.

The analytical solution is similar to that shown in Appendix C.

Numerical Illustrations

Here are illustrative results for the single input switch model, assuming current gross revenue is twice the input cost.

Figure D1 shows that the option coefficient A is positive, β_1 is positive, β_2 is negative, thereby fulfilling the requirements from the theoretical model. The system of value matching conditions, smooth pasting conditions and characteristic root equations is satisfied. When switching is only possible from x to y, assuming $\hat{x}=x=50$ the switching trigger $\hat{y}=12.87$ indicates that y would have to be 25% of the current price before irrevocable switching is justified. The ROV of the switching input opportunity is 534, or more than 71% of the current operating value.

Figure D1

[2] Adkins and Paxson (2011a) allow starting either with x or y, depending on whether x is at least as low as y (in which case, starting with x is logical, if feasible).

	A	B	C	D	E	F	G
1	**Continuous American Perpetual SINGLE SWITCH Option**						
2	ONE WAY SWITCH FROM INPUT x TO y						
3	INPUT x	x	50				
4	INPUT y	y	50				
5	Convenience yield of x	δ_x	0.04				
6	Convenience yield of y	δ_y	0.04				
7	Volatility of x	σ_x	0.40				
8	Volatility of y	σ_y	0.30				
9	Correlation x with y	ρ	0.50				
10	Risk-free interest rate	r	0.05				
11	Output price for x	p_x	100				
12	Output price for y	p_y	100				
13	Switching cost from x to y	S	50				
14							
15	PV of revenues using x	X	750				
16	PV of revenues using y	Y	750				
17	Switching boundary x to y	x	50				
18	NPV	**V_2(x,y)-S**	**700.000**				
19	OPTION VALUE		**534.220**				
20	Asset value in operating mode '1'	**V_1(x,y)**	**1284.220**				
21	Asset value in operating mode '2'	**V_2(x,y)**	**750.000**				
22		A	8.551				
23	Switching boundary x to y	y^(x)	12.869				
24	Solution quadrant	β_1	1.423	*must be positive*			
25	Solution quadrant	β_2	-0.366	*must be negative*			
26		EQUATIONS					
27	Value matching 1		0.000				
28	Smooth pasting 1A		0.000				
29	Smooth pasting 1B		0.000				
30	Solution quadrant 1		0.000				
31		Sum	0.000				
32		SOLVER: SET C31=0, CHANGING C22:C25					
33	Value matching 1	C22*C17^C24*C23^C25-C17/C5+C11/C10+C23/C6-C12/C10+C13					
34	Smooth pasting 1A	C24*C22*C17^(C24-1)*C23^C25-1/C5					
35	Smooth pasting 1B	C25*C22*C17^C24*C23^(C25-1)+1/C6					
36	0.5*C7^2*C24*(C24-1)+0.5*C8^2*C25*(C25-1)+C9*C7*C8*C24*C25+C24*(C10-C5)+C25*(C10-C6)-C10						
37	SPREAD		**37.13**				
38	V_1(x,y)	-C3/C5+C11/C10+C22*C3^C24*C4^C25					
39	V_2(x,y)	-C4/C6+C12/C10					
40	PDE		0.000				
41	ΔROV+1,x		-9.794				
42	ΔROV+1,y		-3.914				
43	ΓROV+1, x		0.129				
44	ΓROV+1,y		0.107				
45	ΓROV+1,x,y		-0.111				
46	0.5*(C7^2)*(C3^2)*C43+0.5*(C8^2)*(C4^2)*C44+C9*C7*C8*C3*C4*C45+(C10-C5)*C3*C41+(C10-C6)*C4*C42-C10*C20+(C11-C3)						
47	ΔROV+1,x	C24*C22*(C3^(C24-1))*(C4^C25)-1/(C5)					
48	ΔROV+1,y	C25*C22*(C3^C24)*(C4^(C25-1))					
49	ΓROV+1, x	C24*(C24-1)*C22*(C3^(C24-2))*(C4^C25)					
50	ΓROV+1,y	C25*(C25-1)*C22*(C3^C24)*(C4^(C25-2))					
51	ΓROV+1,x,y	C24*C25*C22*(C3^(C24-1))*(C4^(C25-1))					

The ROV+ partial derivatives are:

$$\Delta ROV+, x = \beta_1 A x^{\beta_1 - 1} y^{\beta_2} - \frac{1}{\delta_x} \quad \text{(D9)}$$

$$\Delta ROV+, y = \beta_2 A x^{\beta_1} y^{\beta_2 - 1} \quad \text{(D10)}$$

$$\Gamma ROV+, x = \beta_1 (\beta_1 - 1) A x^{\beta_1 - 1} y^{\beta_2} \quad \text{(D11)}$$

$$\Gamma ROV+, y = \beta_2 (\beta_2 - 1) A x^{\beta_1} y^{\beta_2 - 2} \quad \text{(D12)}$$

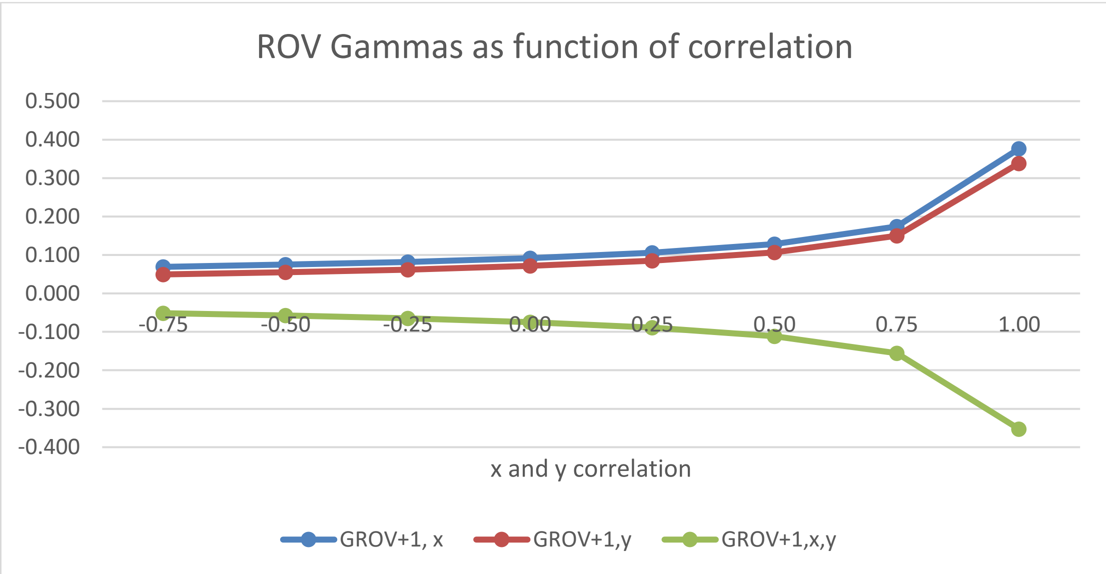
$$\Gamma ROV+, x, y = \beta_1 \beta_2 A x^{\beta_1 - 1} y^{\beta_2 - 1} \quad \text{(D13)}$$

Figure D2 shows that the analytical solution exactly replicates the quasi-analytical numerical solution.

Figure D2

	A	B	C
1	Simplified Støre et al. 2018 Analytical Solution IN-IN Switching		
2	INPUTS		
3	x	50	
4	y	50	
5	δx	0.04	
6	δy	0.04	
7	σx	0.40	
8	σy	0.30	
9	ρ	0.50	
10	r	0.05	
11	px	100	
12	py	100	
13	S	50	
14	X	750	-(B3/B5-B11/B10)
15	Y	750	-(B4/B6-B12/B10)
16	x^	50	B3
17			
18	OUTPUTS		
19	C(x^)	0.960	1-(B5)/(B3)*(B13)
20	f(x^)	0.126	(B7^2)-(2*(B10-B5))-(2*B9*B7*B8)+(B19*(2*(B10-B6)+(B8^2)))
21	g(x^)	0.128	(B7^2)+((B8^2)*(B19^2))-(2*B9*B7*B8*B19)
22	β_1(x^)	1.4232	(B20/(2*B21))+SQRT(((B20/(2*B21))^2)+(2*((B6)/B21)))
23	β_2(x^)	-0.3663	1-B19*B22
24	**y^**	**12.869**	**(-B23*B6*B17)/(B22*B5)**
25	A	8.551	(1/(B22*B5*(B17^(B22-1))*(B24^B23)))
26	ROV	**534.220**	**B25*(B3^B22)*(B4^B23)**
27	VALUE	**1284.220**	**B26-B3/B5+B11/B10**
28	PDE	0.000	
29	ΔROV+1,x	-9.794	B22*B25*(B3^(B22-1))*(B4^B23)-1/(B5)
30	ΔROV+1,y	-3.914	B23*B25*(B3^B22)*(B4^(B23-1))
31	ΓROV+1, x	0.129	B22*(B22-1)*B25*(B3^(B22-2))*(B4^B23)
32	ΓROV+1,y	0.107	B23*(B23-1)*B25*(B3^B22)*(B4^(B23-2))
33	ΓROV+1,x,y	-0.111	B22*B23*B25*(B3^(B22-1))*(B4^(B23-1))
34	V1	1,628.280	B25*(B17^B22)*(B24^B23)-B17/B5+B11/B10
35	V2	1,628.280	-B24/B6+B12/B10-B13
36	0.5*(B7^2)*(B3^2)*B31+0.5*(B8^2)*(B4^2)*B32+B9*B7*B8*B3*B4*B33+(B10-B5)*B3*B29+(B10-B6)*B4*B30-B10*B27+(B11-B3)		
37	ROV	534.220	(B3/(B22*B5))*((B4/B24)^B23)

Figure D3



It is no surprise that the ROV of input switching declines as the correlation of x and y increases as in Figure 12. But perhaps it is less obvious that the ROV gammas (sensitivity of ROV deltas to changes in x or y) increase with correlation, but the cross-gamma decreases. Thus. there is the reoccurring problem of delta and gamma hedging the ROV through dynamic positions in inputs (in this case) without focusing on the correlation of those inputs.